

Unimodular Theory of Gravity and Inflation

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Abstract: We investigate inflation and its scalar perturbation driven by a massive scalar field in the unimodular theory of gravity. We introduce a parameter ξ with which the theory is invariant under general unimodular coordinate transformations. When the unimodular parameter is $\xi = 6$, the classical picture of inflation is reproduced in the unimodular theory because it recovers the background equations of the standard theory of general relativity. We show that for $\xi = 6$, the theory is equivalent to the standard theory of general relativity at the perturbation level. Unimodular gravity constrains the gauge degree of freedom in the scalar perturbation, but the perturbation equations are similar to those in general relativity. For $\xi \neq 6$, we derive the power spectrum and the spectral index, and obtain the unimodular correction to the tensor-to-scalar ratio. Depending on the value of ξ , the correction can either raise or lower the value of the tensor-to-scalar ratio.

1 Introduction

The de Sitter expansion in the period of inflation, provides a solution to the flatness problem, the horizon problem, the entropy problem etc [1–9]. The inflationary model also explains the structure formation of the Universe by considering the perturbations which are generated in the period of inflation. Not only for the period of inflation in the early universe, the role of the exponential expansion of the Universe is important in the current era of the accelerating universe [10, 11]. There are many theories which describe the current accelerating universe such as DGP model, $f(R)$ -gravity and scalar field models, etc. The scalar field model is one of the simplest model. The cosmological constant is also one of the explanation for the current acceleration of the Universe. However, adding the cosmological constant term, the theory suffers from the fine tuning problem.

The unimodular theory of gravity was initially developed in Refs. [12, 13]. One of the motivations of considering the unimodular theory of gravity is to solve the cosmological constant problem [14]. Another interesting implication is that it explains the current expansion of the Universe by considering only single component such as the cosmological constant, or by the non-relativistic matter [15, 16]. The full metric is decomposed in the unimodular metric and a scalar field [15, 16]. The value of the determinant of the unimodular metric is the same as the determinant of the Minkowski metric. The basic idea of the unimodular theory of gravity is to consider the determinant of metric not as a dynamical variable [14, 17–19], and hence the cosmological constant term is absent from the action. However, it was shown in Ref. [14] that the cosmological constant appears as an integration constant in this theory. Some progress in the unimodular theory of gravity has been discussed in Refs. [20–36]. In the case of the unimodular theory we have the unimodular constraint equation, and hence it reduces the gauge degree of freedom. Therefore, the perturbations are determined by the standard equation of motion with the constraint equation.

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In GR, we seek for the gauge invariant scalar perturbation, i.e., the curvature perturbation. This quantity is still an invariant quantity in unimodular gravity since gauge transformations are not related with the determinant of metric.

In the recent paper [31], the authors developed the theory of cosmological perturbations considering the unimodular constraint, i.e., the fixed determinant of the metric. They considered the variation of the metric determinant is zero up to the linear order in the perturbation, and discussed the cosmological perturbations in the matter and the radiation dominated eras. We adopt their method in this paper up to the linear order and get the power spectrum of the cosmological perturbation produced during inflation.

In general, all the metric components are dynamical fields. However, we can assume that some of them are not dynamical [14]. In the unimodular theory, we have the determinant of the metric fixed, i.e., $g_{\mu\nu}\delta g^{\mu\nu} = 0$. All the components are dynamical and adjusted such that they satisfy $\sqrt{-g}$ is fixed. Without any constraint, the gravitational action with the cosmological constant term provides us the standard Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (1)$$

where $R_{\mu\nu}$, R and $T_{\mu\nu}$ are the Ricci tensor, the Ricci scalar, and the energy-momentum tensor of matter. Applying the unimodular constraint, in unimodular gravity we have to subtract all the terms which are proportional to metric $g_{\mu\nu}$ from the Einstein equation. Therefore, we have

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 8\pi G \left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T \right). \quad (2)$$

This is the standard field equation in unimodular gravity. In this paper, we discuss unimodular gravity in a slightly different approach [15, 16]. We first decompose the full metric into the unimodular metric and a scalar field, and write the whole action in terms of these fields with introducing an additional parameter ξ .

The paper is organized as follows. In Sec. 2, we describe the proposed model of the unimodular theory of gravity [15, 16] in brief. In Sec. 3, we derive the background field equations to show that inflation can be reproduced in the unimodular theory in the same way as in GR. In Sec. 4, we discuss the cosmological perturbations. In Sec. 5, we solve the perturbation equation and calculate the power spectrum and the spectral index. In Sec. 6, we conclude.

2 Unimodular Theory of Gravity

We decompose the full metric $g_{\mu\nu}$ into two parts [15, 16] as

$$g_{\mu\nu} = \mathcal{A}^2 \bar{g}_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} = \frac{1}{\mathcal{A}^2} \bar{g}^{\mu\nu}, \quad (3)$$

where $\bar{g}_{\mu\nu}$ and $\bar{g}^{\mu\nu}$ are the metric and the inverse metric corresponding to unimodular gravity. \mathcal{A} is a scalar field which will turn out to be the scale factor of the Universe. The unimodular metric satisfies $\sqrt{-\bar{g}_{\mu\nu}} = f(x)$, where $f(x)$ is the determinant of the Minkowski metric. Using Eq. (3), we can decompose the Christoffel symbol and the Ricci tensor as follows,

$$\Gamma_{\alpha\beta}^{\mu} = \bar{\Gamma}_{\alpha\beta}^{\mu} + \tilde{\Gamma}_{\alpha\beta}^{\mu}, \quad R_{\mu\nu} = \bar{R}_{\mu\nu} + \tilde{R}_{\mu\nu}, \quad (4)$$

where $\bar{\Gamma}_{\alpha\beta}^\mu$ and $\bar{R}_{\mu\nu}$ are computed with the metric $\bar{g}_{\mu\nu}$ and

$$\bar{\Gamma}_{\alpha\beta}^\mu = \bar{g}_\beta^\mu \partial_\alpha \ln \mathcal{A} + \bar{g}_\alpha^\mu \partial_\beta \ln \mathcal{A} - \bar{g}_{\alpha\beta} \partial^\mu \ln \mathcal{A}, \quad (5)$$

$$\begin{aligned} \bar{R}_{\mu\nu} &= -\bar{\Gamma}_{\mu\nu;\alpha}^\alpha + \bar{\Gamma}_{\mu\alpha;\nu}^\alpha - \bar{\Gamma}_{\beta\alpha}^\alpha \bar{\Gamma}_{\mu\nu}^\beta + \bar{\Gamma}_{\beta\nu}^\alpha \bar{\Gamma}_{\mu\alpha}^\beta \\ &= 2(\ln \mathcal{A})_{;\mu\nu} + \bar{g}_{\mu\nu}(\ln \mathcal{A})_{;\alpha}^\alpha - 2\partial_\mu \ln \mathcal{A} \partial_\nu \ln \mathcal{A} + 2\bar{g}_{\mu\nu} \partial_\beta \ln \mathcal{A} \partial^\beta \ln \mathcal{A}. \end{aligned} \quad (6)$$

Using the above definitions, one can also compute the Ricci scalar as

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1}{\mathcal{A}^2} (\bar{R} + \tilde{R}), \quad (7)$$

where

$$\tilde{R} = 6(\ln \mathcal{A})_{;\mu}^\mu + 6\partial_\mu \ln \mathcal{A} \partial^\mu \ln \mathcal{A}. \quad (8)$$

Then the gravitational action can be written as

$$S_E = \int d^4x \sqrt{-g} \frac{1}{16\pi G} [\mathcal{A}^2 \bar{R} + \mathcal{A}^2 \tilde{R}]. \quad (9)$$

In the terms of the scalar field, this can be written as

$$S_E = \int d^4x \sqrt{-g} \frac{1}{16\pi G} [\mathcal{A}^2 \bar{R} - 6\partial_\alpha \mathcal{A} \partial^\alpha \mathcal{A}]. \quad (10)$$

By replacing the coefficient 6 with a new parameter ξ in the action (10), we consider a new action as follows,

$$S_{\text{uni}} = \int d^4x \sqrt{-g} \frac{1}{16\pi G} [\mathcal{A}^2 \bar{R} - \xi \partial_\alpha \mathcal{A} \partial^\alpha \mathcal{A}]. \quad (11)$$

For $\xi = 6$, the theory is equivalent to the standard GR. For $\xi \neq 6$, the action is invariant under the general unimodular coordinate transformations, and \mathcal{A} is treated as a scalar field [15, 16]. Varying the above action with respect to the metric $\bar{g}^{\mu\nu}$ and the field \mathcal{A} , we get the corresponding equations of motion as

$$\begin{aligned} \mathcal{A}^2 \left[\bar{R}_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{R} \right] &+ \left[(\mathcal{A}^2)_{;\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} (\mathcal{A}^2)_{;\lambda}^\lambda \right] - \xi \left[\partial_\mu \mathcal{A} \partial_\nu \mathcal{A} - \frac{1}{4} \bar{g}_{\mu\nu} \partial^\lambda \mathcal{A} \partial_\lambda \mathcal{A} \right] \\ &= -8\pi G \left[T_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} T \right], \end{aligned} \quad (12)$$

and

$$2\mathcal{A}\bar{R} + 2\xi \bar{g}^{\mu\nu} \mathcal{A}_{;\mu\nu} = \kappa T_{\mathcal{A}}, \quad (13)$$

where $\kappa = 16\pi G$, $T_{\mu\nu}$ is the energy-momentum tensor of the matter field, $T = T_\lambda^\lambda$, and $T_{\mathcal{A}}$ includes all the contributions from the coupling of \mathcal{A} with matter fields. In Secs. 3, 4 and 5, we shall derive the physical quantities at the background level as well as at the perturbation level in terms of ξ .

3 Inflation

The equations (12) and (13) have been used to describe the expansion of the Universe for the current era in Refs. [15, 16] in which authors discussed that the expansion of the Universe can be described only by the cosmological constant, or the nonrelativistic matter. In contrast to the standard Λ CDM model, it sounds interesting since only one component of the energy density

can explain the dynamics of the Universe. In this paper, we consider a scalar field ϕ which is responsible for inflation. The matter-field action is given by

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{2} \bar{\mathcal{A}}^2 \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \bar{\mathcal{A}}^4 \phi^2 \right]. \quad (14)$$

Here, we consider the background unimodular metric as $\bar{g}_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. For the above action (14), the energy-momentum tensor of the matter field is given by

$$T_{\mu\nu} = \bar{\mathcal{A}}^2 \partial_\mu \phi \partial_\nu \phi, \quad (15)$$

and $T_{\mathcal{A}}$ is defined as

$$T_{\mathcal{A}} = -\bar{\mathcal{A}} \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2\bar{\mathcal{A}}^3 m^2 \phi^2. \quad (16)$$

For $\xi = 6$, the $(0, 0)$ component of Eq. (12) gives

$$\frac{\mathcal{A}''}{\mathcal{A}} - 2 \left(\frac{\mathcal{A}'}{\mathcal{A}} \right)^2 = -4\pi G \phi'^2, \quad (17)$$

where the prime denotes the derivative with respect to the conformal time η . The equation (13) gives

$$\frac{\mathcal{A}''}{\mathcal{A}} = \frac{8\pi G}{3} \mathcal{A}^2 \left[-\frac{\phi'^2}{2\mathcal{A}^2} + 2V(\phi) \right], \quad (18)$$

where $V(\phi) = m^2 \phi^2 / 2$. Now let us consider $\mathcal{A} = a(t)$, so from Eq. (17) we can write the derivative of the Hubble parameter $H = \dot{a}/a$ as

$$\dot{H} = -4\pi G \dot{\phi}^2, \quad (19)$$

where the overdot denotes the derivative with respect to the cosmological time t defined by the transformation $dt = ad\eta$. From Eqs. (17) and (18), we also recover the first Friedman equation

$$H^2 = \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V \right). \quad (20)$$

The slow-roll parameter ϵ is given by, as usual,

$$\epsilon = -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2}. \quad (21)$$

The Friedmann equations (19) and (20) obtained in unimodular gravity are in the standard form as in GR. The slow-roll parameters are defined also in the same way as in GR. If the kinetic term of the scalar field is much smaller than the potential term, the scale factor exhibits an exponential expansion. The slow-roll parameters determine the sufficient period for inflation which solves the problems of the standard Big Bang cosmology.

Now let us consider the general case of ξ . In this case, from Eq. (12) we have

$$\frac{a''}{a} - \frac{\xi - 2}{2} \left(\frac{a'}{a} \right)^2 = -4\pi G \phi'^2. \quad (22)$$

If we introduce $\delta \equiv \xi - 6$, with δ of an order of unity, we have

$$\dot{H} = \frac{\delta}{2} H^2 - 4\pi G \dot{\phi}^2. \quad (23)$$

The equation (13) gives

$$\xi \frac{a''}{a} = \kappa a^3 \left[-\frac{\phi'^2}{2a^2} + 2V(\phi) \right]. \quad (24)$$

Using Eqs. (22) and (24), we have

$$\begin{aligned} H^2 &= \frac{64\pi G}{\xi(\xi-2)} \left[\frac{(\xi-2)}{4} \frac{\dot{\phi}^2}{2} + V \right] \approx \frac{8\pi G}{3} \left[\left(1 + \frac{\delta}{4}\right) \frac{\dot{\phi}^2}{2} + V \right] \left(1 + \frac{\delta}{6}\right)^{-1} \left(1 + \frac{\delta}{4}\right)^{-1} \\ &\approx \left(1 + \frac{\delta}{6}\right)^{-1} \left(1 + \frac{\delta}{4}\right)^{-1} H_{\text{GR}}^2, \end{aligned} \quad (25)$$

where the last approximation is for the slow-roll region, and $H_{\text{GR}} \approx \sqrt{(8\pi G)V/3}$ is the Hubble parameter in GR. The form of equation of motion for ϕ is as usual since the parameter ξ is only used in the gravitational action. The slow-roll parameter becomes

$$\epsilon = -\frac{\dot{H}}{H^2} = \epsilon_1 - \frac{\delta}{2}, \quad (26)$$

where

$$\epsilon_1 = 4\pi G \frac{\dot{\phi}^2}{H^2} \approx \frac{1}{16\pi G} \left(\frac{V'}{V} \right)^2 \left(1 + \frac{\delta}{6}\right)^2 \left(1 + \frac{\delta}{4}\right)^2 = \frac{2M_p^2}{\phi^2} \left(1 + \frac{\delta}{6}\right)^2 \left(1 + \frac{\delta}{4}\right)^2. \quad (27)$$

For $\xi = 6$ ($\delta = 0$), the slow-roll parameter $\epsilon_1 = 2M_p^2/\phi^2$ is the same as in the standard GR. The parameter δ modifies the slow-roll parameter ϵ via Eq. (26), and thus the number of e -foldings. The Hubble parameter is also slightly modified. The negative small value of δ increases the value of Hubble parameter. Using Eq. (26), one can write,

$$\frac{\ddot{a}}{a} = \left(1 + \frac{\delta}{2} - \epsilon_1\right) H^2. \quad (28)$$

From this, the necessary condition for inflation $\ddot{a} > 0$ puts a lower bound on the parameter, $\delta > -2$ ($\xi > 4$).

4 Cosmological Perturbations

In this section, we discuss the cosmological perturbations. The decomposition of the perturbations into the scalar, the vector and the tensor fields is still valid in the unimodular theory. We consider only the scalar perturbations in this paper. The perturbed metric and scalar field are defined as

$$ds_g^2 = a^2 \left\{ (-1 - 2A) d\eta^2 + 2B_{,i} dx^i d\eta + \left[(1 - 2\psi) \delta_{il} + D_{il} E \right] dx^i dx^l \right\} \equiv a^2 ds_g^2, \quad (29)$$

$$\phi = \chi_0(\eta) + \chi_1(\eta, \vec{x}), \quad (30)$$

where A, ψ, E and B are scalar perturbation fields, $D_{ij} \equiv \partial_i \partial_j - (1/3) \delta_{ij} \nabla^2$, and ds_g is the physical line element. We perturb each term in Eq. (12) in the linear order, and write all the components,

$(0,0)$, $(i,0)$ and (i,j) of these terms. In the unimodular theory $\sqrt{-\bar{g}}$ is not dynamical, but the line element which is governed by the full metric $g_{\mu\nu}$, is still invariant under general coordinate transformations. Therefore, we have the gauge dependent gravitational potential, ψ , A , etc., same as in GR. In the linear order, the constraint $\delta\sqrt{-g} = 0$ gives

$$A - 3\psi + E_{,ii} = 0. \quad (31)$$

For the background, $\bar{R}_{\mu\nu}$ and $\bar{R} = 0$, since the background metric $\bar{g}_{\mu\nu}$ is the Minkowski metric. Only the perturbed part of these can survive. The $(0,0)$, (i,j) and $(0,i)$ components of the perturbed part of \mathcal{A}^2 ($\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}$) are given by

$$\mathcal{A}^2 \left(\bar{R}_{00} - \frac{1}{4}\bar{g}_{00}\bar{R} \right) = \frac{\mathcal{A}^2}{2} \left[(A + B' + 2\psi)_{,i}{}^i + 3\psi'' + \frac{1}{2}(D_i^k E)_{,k}{}^i \right], \quad (32)$$

$$\begin{aligned} \mathcal{A}^2 \left(\bar{R}_{ij} - \frac{1}{4}\bar{g}_{ij}\bar{R} \right) &= \mathcal{A}^2 \left[\delta_{ij} \left(\frac{\psi''}{2} + \frac{1}{2}(B')_{,l}{}^l + \frac{1}{2}A_{,l}{}^l - \frac{1}{4}(D_l^k E)_{,k}{}^l \right) - B'_{,ij} + \psi_{,ij} \right. \\ &\quad \left. - A_{,ij} + \frac{1}{2}D_{ij}E'' + \frac{1}{2}(D_j^k E)_{,ik} + \frac{1}{2}(D_i^k E)_{,jk} - \frac{1}{2}(D_{ij})_{,k}{}^k E \right], \end{aligned} \quad (33)$$

$$\mathcal{A}^2 \left(\bar{R}_{i0} - \frac{1}{4}\bar{g}_{i0}\bar{R} \right) = \mathcal{A}^2 \left(2\psi'_{,i} + \frac{1}{2}(D_i^k E')_{,k} \right). \quad (34)$$

The $(0,0)$, (i,j) and $(i,0)$ components of the perturbed part of other terms, $(\mathcal{A}^2)_{;\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}(\mathcal{A}^2)_{;\lambda}{}^\lambda$, $\partial_\mu \mathcal{A} \partial_\nu A - \frac{1}{4}\bar{g}_{\mu\nu} \partial^\lambda \partial_\lambda \mathcal{A}$, and $T_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu} T$, are given by

$$\begin{aligned} (\mathcal{A}^2)_{;00} - \frac{1}{4}\bar{g}_{00}(\mathcal{A}^2)_{;\lambda}{}^\lambda &= -2\mathcal{A}\mathcal{A}'A' + \frac{\mathcal{A}\mathcal{A}'}{2} \left(A' + B_{,i}{}^i + 3\psi' - \frac{1}{2}D_i^i E' \right) \\ &= \frac{\mathcal{A}\mathcal{A}'}{2} \left(-3A' + B_{,i}{}^i + 3\psi' - \frac{1}{2}D_i^i E' \right), \end{aligned} \quad (35)$$

$$\begin{aligned} (\mathcal{A}^2)_{;ij} - \frac{1}{4}\bar{g}_{ij}(\mathcal{A}^2)_{;\lambda}{}^\lambda &= 2\mathcal{A}\mathcal{A}' \left(B_{,ij} + \delta_{ij}\psi' - \frac{1}{2}D_{ij}E' \right) + \frac{1}{2}(\mathcal{A}'^2 + \mathcal{A}\mathcal{A}'')D_{ij}E \\ &\quad - \delta_{ij} \left[(\mathcal{A}'^2 + \mathcal{A}\mathcal{A}'')(\psi + A) + \frac{\mathcal{A}\mathcal{A}'}{2} \left(A' + B_{,l}{}^l + 3\psi' - \frac{1}{2}D_k^k E' \right) \right], \end{aligned} \quad (36)$$

$$(\mathcal{A}^2)_{;i0} - \frac{1}{4}\bar{g}_{i0}(\mathcal{A}^2)_{;\lambda}{}^\lambda = -2\mathcal{A}\mathcal{A}'A_{,i} + \frac{1}{2}(\mathcal{A}'^2 + \mathcal{A}\mathcal{A}'')B_{,i}, \quad (37)$$

$$\partial_0 \mathcal{A} \partial_0 A - \frac{1}{4}\bar{g}_{00} \partial^\lambda \partial_\lambda \mathcal{A} = 0, \quad (38)$$

$$\partial_i \mathcal{A} \partial_j A - \frac{1}{4}\bar{g}_{ij} \partial^\lambda \partial_\lambda \mathcal{A} = -\frac{1}{2}\mathcal{A}'^2 \left[(\psi + A)\delta_{ij} - \frac{1}{2}D_{ij}E \right], \quad (39)$$

$$\partial_i \mathcal{A} \partial_0 A - \frac{1}{4}\bar{g}_{i0} \partial^\lambda \partial_\lambda \mathcal{A} = \frac{\mathcal{A}'^2}{4}B_{,i}, \quad (40)$$

$$T_{00} - \frac{1}{4}\bar{g}_{00}T = \frac{3}{2}\mathcal{A}^2\chi'_0\chi'_1, \quad (41)$$

$$T_{ij} - \frac{1}{4}\bar{g}_{ij}T = \delta_{ij} \frac{\mathcal{A}^2}{2} (-\chi_0'^2\psi - \chi_0'^2A + \chi_0'\chi_1') + \frac{\mathcal{A}^2\chi_0'^2}{4}D_{ij}E, \quad (42)$$

$$T_{i0} - \frac{1}{4}\bar{g}_{i0}T = \mathcal{A}^2 \left(\chi_0'\chi_{1,i} + \frac{1}{4}\chi_0'^2B_{,i} \right). \quad (43)$$

Varying the matter action (14) with respect to the scalar field, we have

$$\bar{D}^\mu \bar{D}_\mu \phi + 2 \frac{\partial^\mu \mathcal{A}}{A} \partial_\mu \phi - m^2 \mathcal{A}^2 \phi = 0. \quad (44)$$

Using $\mathcal{A} = a(t)$, we obtain the zeroth and the first order equations as

$$\chi_0'' + \frac{2a'}{a} \chi_0' + m^2 a^2 \chi_0 = 0, \quad (45)$$

$$\chi_1''(k, \eta) + \frac{2a'}{a} \chi_1'(k, \eta) - 2A(\eta, k) \left(2 \frac{a'}{a} \chi_0' + \chi_0'' \right) + m^2 a^2 \chi_1(k, \eta) + k^2 \chi_1(k, \eta) = 0, \quad (46)$$

where we expressed the perturbations in terms of the fourier mode such as

$$\chi_1(\eta, \vec{x}) = \int \frac{d^3 k}{(2\pi)^{3/2}} \chi_1(\eta, \vec{k}) e^{i\vec{k} \cdot \vec{x}}. \quad (47)$$

Using Eq. (45), we can simplify Eq. (46) as

$$\chi_1'' + \frac{2a'}{a} \chi_1' + k^2 \chi_1 + m^2 a^2 (2\chi_0 A + \chi_1) = 0. \quad (48)$$

This equation is slightly different from the standard one because of the constraint on $\sqrt{-g}$. The constraint reduces the gauge degree of freedom. In our case, we cannot set both E and B to zero. We can set only one of these fields to zero since we have the constraint Eq. (31). The comoving curvature is still a relevant quantity since it is independent of E and B . In the next section, we solve for the power spectrum of the scalar perturbation choosing a gauge $E = 0$ ($B \neq 0$).

5 Power Spectrum

In this section, we solve Eq. (48) for the scalar perturbation. In the gauge $E = 0$, we have $A = 3\psi$ from Eq. (31). Using Eqs. (32)-(43), we can write all the components of Eq. (12). The (i, j) ($i \neq j$) and $(0, i)$ components of Eq. (12) are useful in eliminating the field A in Eq. (48) and can be written as

$$\psi - A = B' - 2\mathcal{H}B, \quad (49)$$

$$\psi' - \mathcal{H}A + \frac{1}{4} \left[\mathcal{H}^2 \left(1 - \frac{\xi}{2} \right) + \frac{a''}{a} + 4\pi G \chi_0'^2 \right] B = -4\pi G \chi_0' \chi_1, \quad (50)$$

where, $\mathcal{H} = a'/a$. Using Eqs. (22) and (50), we have

$$A \approx \frac{4\pi G \chi_0'}{\mathcal{H}} \chi_1, \quad (51)$$

where we assume that the gravitational potential does change much ($\psi' \approx 0$) in the super-horizon scale. Plugging A in Eq. (51) into Eq. (48), we obtain

$$\chi_1'' + 2 \frac{a''}{a} \chi_1' + k^2 \chi_1 + m^2 a^2 (1 + s) \chi_1 \approx 0, \quad (52)$$

where $s = 2\chi_0\sqrt{4\pi G\epsilon_1} \ll 1$. Introducing $\sigma = a\chi_1$, we can rewrite Eq. (52) as

$$\sigma'' + \left[k^2 - \frac{a''}{a} + (1+s)m^2a^2 \right] \sigma \approx 0. \quad (53)$$

In the quasi de-Sitter expansion ($\dot{H} = -\epsilon H^2$), $a(\eta) = -1/(1-\epsilon)H\eta$, one can have

$$\frac{a''}{a} = a^2 \left(2H^2 + \dot{H} \right) \approx \frac{2+3\epsilon}{\eta^2}, \quad (54)$$

and,

$$\begin{aligned} -\frac{a''}{a} + (1+s)m^2a^2 &\approx -\frac{2+3\epsilon}{\eta^2} + (1+s)(1+2\epsilon)\frac{m^2}{H^2\eta^2} \\ &\approx -\frac{2+3\epsilon-3\eta_\chi}{\eta^2} = -\left(\nu_\chi^2 - \frac{1}{4}\right)\frac{1}{\eta^2}, \end{aligned} \quad (55)$$

where $\eta_\chi = m^2/3H^2 \ll 1$ and $\nu_\chi = 3/2 + \epsilon - \eta_\chi$. Therefore, Eq. (53) can be expressed in the following form,

$$\sigma'' + \left[k^2 - \left(\nu_\chi^2 - \frac{1}{4}\right)\frac{1}{\eta^2} \right] \sigma \approx 0. \quad (56)$$

The solution to this equation is given by,

$$\sigma = \sqrt{-\eta} \left[c_1(k)H^{(1)}(-k\eta) + c_2(k)H^{(2)}(-k\eta) \right], \quad (57)$$

where $H^{(1,2)}$ is the Hankel function of first and second kind. In the ultraviolet regime, $k \gg aH(-k\eta \gg 1)$, we have

$$H^{(1)}(-k\eta \gg 1) \approx \sqrt{-\frac{2}{\pi k\eta}} e^{i(-k\eta - \pi\nu_\chi/2 - \pi/4)}, \quad H^{(2)}(-k\eta \gg 1) \approx \sqrt{-\frac{2}{\pi k\eta}} e^{-i(-k\eta - \pi\nu_\chi/2 - \pi/4)}. \quad (58)$$

After imposing the boundary condition $c_2(k) = 0$ and $c_1(k) = (\sqrt{\pi}/2) e^{i(\pi\nu_\chi/2 + \pi/4)}$, the solution becomes a plane wave $e^{-ik\eta}/\sqrt{2k}$. Then we have

$$\chi_1 \approx \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH} \right)^{3/2 - \nu_\chi}, \quad (59)$$

and the spectral index n_s is given by

$$n_s = 1 + \frac{d \ln(P_\chi)}{d \ln(k)} = 1 + 2\eta_\chi - 2\epsilon. \quad (60)$$

The comoving curvature $\mathcal{R} = \psi + (\mathcal{H}\chi_1)/\chi'_0 = A/3 + (\mathcal{H}\chi_1)/\chi'_0 \approx (\mathcal{H}\chi_1)/\chi'_0$ is still a gauge invariant quantity since all the gauge transformations of perturbations which make the line element

ds_g invariant, are not related with $\sqrt{-g}$. The power spectrum is given by

$$\begin{aligned}
 P_{\mathcal{R}} = \frac{k^3}{2\pi^2} \mathcal{R}^2 &\approx \frac{k^3}{2\pi^2} \frac{H^2 \chi_1^2}{\dot{\chi}_0^2} \approx \frac{k^3 H^2}{2\pi^2 \dot{\chi}_0^2} \times \frac{H^2}{2k^3} \left(\frac{k}{aH} \right)^{3-2\nu_\chi} \\
 &\approx \frac{H^2}{\dot{\chi}_0^2} \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{3-2\nu_\chi} \approx \frac{4\pi G}{\epsilon_1} \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{3-2\nu_\chi} \\
 &\approx \frac{4\pi G}{\epsilon_1} \left(\frac{H}{2\pi} \right)^2.
 \end{aligned} \tag{61}$$

Here, we note that the observed curvature perturbation is dependent of ϵ_1 which is a function of the parameter δ . The action corresponding to the tensor perturbation is independent of the parameter ξ . Due to the traceless property of the tensor perturbation, $\sqrt{-g}$ is fixed up to the linear order, and thus it does not affect the equation of the tensor perturbation up to the linear order. Therefore, the tensor power spectrum is the same as in the standard one,

$$P_T = 64\pi G \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{3-2\nu_T} \approx 64\pi G \left(\frac{H}{2\pi} \right)^2, \tag{62}$$

where $\nu_T \sim 3/2$. The tensor-to-scalar ratio r is then given by

$$r = \frac{P_T}{P_{\mathcal{R}}} \approx 16\epsilon_1 \approx 32 \left(1 + \frac{\delta}{6} \right)^2 \left(1 + \frac{\delta}{4} \right)^2 \frac{M_P^2}{\chi_0^2} \approx \left(1 + \frac{\delta}{6} \right)^2 \left(1 + \frac{\delta}{4} \right)^2 r_{\text{GR}}, \tag{63}$$

where $r_{\text{GR}} \sim 0.131$ is the tensor-to-scalar ratio in GR [$\xi = 6$ ($\delta = 0$)]. One can have $r < r_{\text{GR}}$ when $-10 \leq \delta \leq 0$. Requiring $H^2 > 0$ and $\ddot{a} > 0$ constrains the parameter further as $\delta > -2$ ($\xi > 4$). In order to have a sufficient period of inflation, we consider $|\delta| < 1$. From the recent analyses of Planck Collaboration [37], the value is expected as $r \leq 0.09$ which puts an upper bound on parameter as $\delta \lesssim -0.43$ ($\xi \lesssim 5.57$).

6 Conclusions

In this paper, we have shown that classical picture of inflation can be reproduced by the unimodular theory of gravity, and have developed it in the perspective of the cosmological perturbation during inflation. The unimodular theory is a subspace of the general theory of relativity. It reduces the number of the gauge degree of freedom. We have considered a gauge condition $E = 0$ ($B \neq 0$) which is similar to the standard longitudinal gauge $E = B = 0$. We obtained the similar perturbation relation as in the standard case. The tensor perturbation is not altered in this theory since the standard tensor perturbation fortunately satisfies the constraint $\sqrt{-g}$ fixed up to the linear order due to its traceless property, and also it is independent of the unimodular parameter ξ .

We have shown that for $\xi = 6$, all the physical quantities at the background level as well as at the perturbation level are the same as the standard GR. For $\xi \neq 6$, the number of e -foldings is modified at the same scale of inflaton field. The value of the Hubble parameter increases as the ξ decreases from 6. The tensor-to-scalar ratio $r = 16\epsilon_1$ can be lowered simultaneously. We found from the background equation that the physical accepted range is $4 < \xi \leq 6$. For the range of the tensor-to-scalar ratio $r \leq 0.09$ from the new analyses of Planck collaboration [37], we have found upper bound on ξ is ~ 5.57 .

The enhanced gravitational wave has been observed by BICEP2 collaboration [38]. Recently,

Planck collaboration suggested the possibility of the enhanced signal is due to dust contaminations [37], but it still needs to be confirmed by new experiments. Since we introduced an additional parameter ξ in this work, fitting the values of the tensor-to-scalar ratio and the spectral index is more flexible. This value of ξ will be further constrained by forthcoming observations. Detailed study of the scalar and the tensor perturbation up to the second order will be pursued in the future publication.

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